Math 564: Real analysis and measure theory
Lecture 19

Fubini-Tonelli for sets. Let $(X, \mathcal{I}, \mu)$ and $(Y, \mathcal{J}, \nu)$ be σ-finite measure spaces. Let $R \subseteq \mathcal{I} \otimes \mathcal{J}$ (i.e. $\mathbb{1}_R$ is $\mathcal{I} \otimes \mathcal{J}$-measurable). Then:

(i) The functions $x \mapsto \nu(R_x)$ and $y \mapsto \mu(R^y)$ are $\mathcal{I}$- and $\mathcal{J}$-measurable, respectively.

(ii) $\int_X \nu(R_x) \, d\mu(x) = \mu \times \nu(R) = \int_Y \mu(R^y) \, d\nu(y)$.

Proof. Firstly, we may assume WLOG that μ, ν are finite by the usual argument of writing $X = \bigsqcup_{n \in \mathbb{N}} X_n$ and $Y = \bigsqcup_{m \in \mathbb{N}} Y_m$ for some $X_n \in \mathcal{I}$, $Y_m \in \mathcal{J}$ of finite measure so $X \times Y = \bigsqcup_{n,m \in \mathbb{N}} X_n \times Y_m$ and $R = \bigsqcup_{n,m \in \mathbb{N}} (R \cap X_n \times Y_m)$, and using closure under limits for (i) and ctbl additivity for (ii).

Let $\mathcal{C} := \{R \in \mathcal{I} \otimes \mathcal{J} : R$ satisfies (i) and (ii)$\}$ and aim to show that $\mathcal{C}$ is a σ-algebra containing the algebra $\mathcal{A}$ generated by rectangles $I \times J$ with $I \in \mathcal{I}$, $J \in \mathcal{J}$.

$\mathcal{C}$ contains rectangles: Indeed, for $I \in \mathcal{I}$ and $J \in \mathcal{J}$, the function $x \mapsto \nu((I \times J)_x) = \nu(J) \cdot \mathbb{1}_I(x)$ which is clearly measurable, and same for the y-fibers. For (ii), observe that $\int_X \nu((I \times J)_x) \, d\mu(x) = \int_X \nu(J) \cdot \mathbb{1}_I(x) \, d\mu(x) = \nu(J) \cdot \mu(I) = \mu \times \nu(I \times J)$, and same for y-fibers.

$\mathcal{C}$ is closed under disjoint unions: This is because the measures are finitely additive, measurable functions are closed under addition and integral is linear.

Thus, $\mathcal{C}$ contains the algebra $\mathcal{A}$ because each element of $\mathcal{A}$ is a finite disjoint union of rectangles.

Using closedness under limits of measurable functions and MCT, we also get that $\mathcal{C}$ is closed under ctbl disjoint unions, and using finiteness of μ, ν, and $\mu \times \nu$, we can also deduce that $\mathcal{C}$ is closed under complements, e.g. $\mu \times \nu(R^c) = \mu \times \nu(X \times Y)$

$-\mu\times\nu(R)$. But to conclude that $\mathcal{C}$ is a σ-algebra, we still need to show closedness under finite intersections (to disjointify a ctbl union, this is needed), which is hard to show because measures the measure of the intersection is not expressible by the measures of the sets. Instead, we appeal to the monotone class lemma, to be proved below, and see that is enough to verify that $\mathcal{C}$ is closed under ctbl increasing unions and ctbl decreasing intersections, i.e. is a monotone class, because then, $\mathcal{C} \supseteq \langle \mathcal{A} \rangle_\sigma = \mathcal{I} \otimes \mathcal{J}$, hence $\mathcal{C} = \mathcal{I} \otimes \mathcal{J}$.

$\mathcal{C}$ is a monotone class: For (i) use the monotone converge properties of measures (including the decreasing convergence due to finiteness of measures) and closedness of measurable functions under pointwise limits. For (ii), for increasing unions apply MCT, and for decreasing intersections, apply DCT.

Def. A collection $\mathcal{C}$ of subsets of a set X is called a monotone class if it is closed under ctbl increasing unions and ctbl decreasing intersections. The monotone class generated by a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is the $\subseteq$-least monotone class containing $\mathcal{A}$, i.e. the intersection of all monotone classes containing $\mathcal{A}$.

Monotone Class Lemma. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a monotone class. If $\mathcal{C}$ contains an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ then $\mathcal{C} \supseteq \langle \mathcal{A} \rangle_\sigma$.

Proof. By shrinking $\mathcal{C}$, we may assume WLOG that $\mathcal{C}$ is the monotone class generated by $\mathcal{A}$. Then we show that $\mathcal{C} = \langle \mathcal{A} \rangle_\sigma$. For this we need to show that $\mathcal{C}$ is closed under complements and ctbl unions, but a ctbl union $\bigcup C_n = \bigcup\limits^{\uparrow}_{n\in\mathbb{N}} \left(\bigcup\limits_{m\leq n} C_m\right)$ and $\mathcal{C}$ is closed under ctbl increasing unions, so it is enough to show that $\mathcal{C}$ is closed under finite unions and complements, i.e. is an algebra.

Complements: Let $\mathcal{S} := \{S \in \mathcal{C} : S^c \in \mathcal{C}\}$ and show that $\mathcal{S} \supseteq \mathcal{A}$ and is a monotone class. But for $A \in \mathcal{A}$, A^c is also in $\mathcal{A}$, so $A \in \mathcal{S}$. As for ctbl incr. unions and ctbl decr. inters., observe that $\left(\bigcup\limits^{\uparrow}_{n\in\mathbb{N}} S_n\right)^c = \bigcap\limits^{\downarrow}_{n\in\mathbb{N}} S_n^c$ and $\left(\bigcap\limits^{\downarrow}_{n\in\mathbb{N}} S_n\right)^c = \bigcup\limits^{\uparrow}_{n\in\mathbb{N}} S_n^c$ so $\mathcal{C}$ being closed under these operations implies that so is $\mathcal{S}$. Hence $\mathcal{S} = \mathcal{C}$.

Finite unions. For each $U \in \mathcal{C}$, let $\mathcal{S}(U) := \{V \in \mathcal{C} : U \cup V \in \mathcal{C}\}$. We need to show that for each $U \in \mathcal{C}$, the collection $\mathcal{S}(U) \supseteq \mathcal{A}$ and is a monot. class, because then $\mathcal{S}(U) = \mathcal{C}$ hence $\mathcal{C}$ is closed under finite unions. To check that $\mathcal{S}(U)$ is a monotone class, suppose $\{V_n\} \subseteq \mathcal{S}(U)$ and is increasing, and observe that $U \cup \bigcup\limits^{\uparrow}_{n\in\mathbb{N}} V_n = \bigcup\limits^{\uparrow}_{n\in\mathbb{N}} (U \cup V_n)$ hence it is in $\mathcal{S}(U)$ because $\mathcal{C}$ is closed under ctbl increasing unions. Now suppose $\{V_n\} \subseteq \mathcal{S}(U)$ and is decreasing, then $U \cup \bigcap\limits^{\downarrow}_{n\in\mathbb{N}} V_n = \bigcap\limits^{\downarrow}_{n\in\mathbb{N}} (U \cup V_n)$ and the latter is in $\mathcal{C}$ since $\mathcal{C}$ is closed under ctbl decreasing intersections.
Finally, to show that $\mathcal{A} \subseteq \mathcal{S}(U)$ we fix $A \in \mathcal{A}$ and show $A \in \mathcal{S}(U)$. But the latter is equivalent to $U \in \mathcal{S}(A)$. We in fact show that $\mathcal{S}(A) = \mathcal{C}$ hence $U \in \mathcal{S}(A)$. We already that $\mathcal{S}(A)$ is a monotone class (we proved above for all $U \in \mathcal{C}$). Also $\mathcal{A} \subseteq \mathcal{S}(A)$ because $\mathcal{A}$ is closed under finite unions and $\mathcal{C} \supseteq \mathcal{A}$. Thus, $\mathcal{S}(A)$ contains the monotone class generated by $\mathcal{A}$, hence $\mathcal{S}(A) = \mathcal{C}$. $\square$

Theorem (Fubini-Tonelli for $\mathcal{I} \otimes \mathcal{J}$). Let $(X, \mathcal{I}, \mu)$ and $(Y, \mathcal{J}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to \overline{\mathbb{R}} := [-\infty, \infty]$ be a $\mathcal{I} \otimes \mathcal{J}$-measurable function. Then:

(a) $f_x : Y \to \overline{\mathbb{R}}$ and $f^y : X \to \overline{\mathbb{R}}$ are $\mathcal{J}$ and $\mathcal{I}$-measurable for all $x \in X$ and $y \in Y$

(b) Tonelli. If $f \geq 0$, then:

(i) $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are $\mathcal{I}$ and $\mathcal{J}$-measurable.

(ii) $$\int_X \int_Y f_x(y) \, d\nu(y) \, d\mu(x) = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \int_X f^y(x) \, d\mu(x) \, d\nu(y).$$

(c) Fubini. If f is $\mu\times\nu$-integrable then:

(i) $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are $\mathcal{I}$ and $\mathcal{J}$-measurable and integrable.

(ii) $$\int_X \int_Y f_x(y) \, d\nu(y) \, d\mu(x) = \int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \int_X f^y(x) \, d\mu(x) \, d\nu(y).$$

Proof. We have already proved (a) and we know (b) for indicator functions, which implies (b) and (c) for simple functions by the linearity of integral.
To conclude (b) for all $f \geq 0$, write f as an increasing limit of nonnegative simple functions and in (i) use the closedness of measurable functions under ptwise limit and MCT. For (ii), just use MCT three times. Finally, for (c), write $f = f_+ - f_-$, so f_+ and f_- are $\mu \times \nu$-integrable and apply (b) to f_+ and f_- individually, observing that the finiteness of $\int f_\pm \, d\mu\times\nu$ implies that the functions $x \mapsto \int f_x \, d\nu$ and $y \mapsto \int f^y \, d\mu$ are finite a.e. Get (c) for f by linearity of integral. $\square$

The $\mu\times\nu$-measurable version of this theorem follows for the $\mathcal{I}\otimes\mathcal{J}$-measurable version and is left as a HW exercise.